Meaningful Learning Experience for Problem Solving and Mathematics Competition Questions

Toh Tin Lam

Abstract

It has been argued that teachers and educators should view the role of mathematics competition beyond merely identifying mathematical "talents" or nurturing high achievers in mathematics. Many of the mathematics competition questions are useful even in the usual mathematics classrooms for the general student population. In this note the author argues that many competition questions could provide valuable learning experiences for students to learn mathematical problem solving. Two questions from the Singapore Mathematical Olympiad are selected for discussion. To enhance students' meaningful learning for problem solving, teachers could use appropriate scaffolding to facilitate their students' learning. The scaffolding selected is based on the theoretical framework of Polya's problem solving model.

Learning Experience in Mathematics

It is a truism that not only what students learn is important; how students learn is at least as important. It is thus not surprising to see that in the recent revision of the mathematics curriculum, the emphasis on the learning experiences is shown in the mathematics curriculum document. According to the syllabus document, one of the aims of the Singapore mathematics curriculum is to ensure that "[t]hese processes [i.e. the cognitive and the metacognitive skills] are learned through carefully constructed learning experiences." (Ministry of Education, 2013).

These statements about learning experiences, which are expressed in the form of "students should have opportunities to…", serve to remind teachers of the student-centric nature of these experiences (Ministry of Education, 2013) and, more importantly, the essence of learning mathematics. Teachers should then reflect on the essence of teaching and learning mathematics. Does teaching and learning mathematics consist solely of formulae and procedures required for solving examination questions? Is there more to teaching and learning for assessments or examinations? The answer to such questions is not too far away if one examines the syllabus document carefully: problem solving is the heart of the Singapore mathematics curriculum.

Problem Solving

Mathematical problem solving ability refers to an individual's ability to solve a non-routine mathematical task for which the solution is not immediately obvious to the individual. As such, one needs to learn the approaches to handle non-routine problems without solely depending on formulae or procedures which are acquired instrumentally. Such approaches to handle non-routine problems are usually summed up as a "problem solving model". In the Singapore mathematics curriculum, Polya's problem solving model was officially documented in the syllabus document (although any other sound problem solving model will be equally viable). A brief description of the four stages of Polya's problem solving model can be found in Toh, Quek, Leong, Dindyal & Tay (2011), which we shall not discuss in details here.

Good mathematics students would presumably have built up their own models of problem solving. Most mathematics teachers and educators will generally agree that a student learning mathematics requires a problem solving model to which he or she can depend on, especially when progress in solving a particular mathematics problem is not satisfactory.

Thus, it would not be too far-fetched if one were to conclude that in teaching students mathematics according to the truest spirit of the Singapore mathematics curriculum, every student must be engaged in meaningful learning experiences that contribute to the students developing their competency in problem solving. We shall not first identify what this "meaningful learning experience" represents, but describe what this should not be: anecdotal evidence from the classrooms generally suggests that instead of teaching problem solving, some teachers are routinizing mathematical problem solving into sets of formulae and procedures (or "heuristics") for students to memorize. This approach might be economical to some extent in handling high-stake national examinations. However, it definitely defies the spirit of true problem solving.

In identifying meaningful learning experience for problem solving that is necessary for students, I believe that mathematics competition questions hold promise in this direction. In the remaining section of this paper, we will describe with two examples how meaningful learning experience for problem solving can be enhanced in handling several mathematics competition questions.

Mathematics Competition Questions and Problem Solving

Some educators and mathematicians are beginning to use mathematics competition questions beyond the mere objective of identifying and developing the mathematically talented for the IMO or other mathematics competitions. Many mathematics competition questions (especially those at regional level) are not beyond the reach of the general student population. According to the Singapore Mathematical Society, for example, besides identifying talents in mathematics (this is always one of the aims of any mathematics competition), one of the main objectives of the Singapore Mathematical Olympiad is to "enhance students' [here it refers to the general student population, and not the high achievers alone] problem-solving skills that they learnt in school" (Tay, To, Toh & Wang, 2011).

Exposing students to mathematical problem solving via mathematics competition questions is not solely a theoretical talk. Holton (2010), through imparting the mathematical content knowledge required for the International Mathematical Olympiad (IMO) on discrete mathematics to the New Zealand IMO team, introduced mathematical problem solving processes (which he believes to be the essence of mathematics) and used scaffoldings by providing hints on the use of appropriate problem solving heuristics for selected IMO problems on discrete mathematics. Motivated by this approach, Toh, Quek, Leong, Dindyal and Tay (2011) developed a problem solving module, which emphasizes the processes of problem solving, for secondary school students. The module consists of seventeen problems, which were each selected to highlight different aspects of problem solving to be introduced at different junctures. Many of these problems were adapted or modified from various mathematics competition questions (for the complete collection of the problems, readers may refer to the website http://math.nie.edu.sg/mprose).

There are at least three reasons why many competition questions can provide meaningful learning experiences essential for problem solving, compared to the routine questions from school textbooks. Firstly, these questions are generally difficult for most students (and even for teachers!). The solutions to these questions are usually not easy to obtain if one relies solely on applying formulae or standard procedure; solving these problems calls for experiences beyond mere recall or application. Here, a skilful teacher can demonstrate the importance of competent problem solving skills in addition to correct application of formulae or procedures.

Secondly, the creative nature of many of these problems makes it difficult for one to attempt to routinize these problems in such a way that the method that can be memorized is applicable to other problems one is likely to encounter.

Thirdly, many of these questions are mathematically rich to be used for students to "look back" at their solution - the fourth stage of the Polya's problem solving model. By reconsidering and re-examining their own solution or the mathematics problem, students can consolidate their knowledge and develop their own ability to solve problems (Polya, 1945). "Looking back" at a solved mathematical problem could also include comparing alternative solutions, posing alternative problems and making generalizations (Cai & Brook, 2006).

Mathematics Competition Questions from Previous Years

We shall next demonstrate with two examples of mathematics competition questions selected from the past years mathematics competition (in the Singapore Mathematical Olympiad or SMO) on how rich mathematical learning can be acquired by engaging students with these tasks. Readers who are interested in the problems are urged to solve these questions themselves. Official solutions of these problems are available in the SMO Problem and Solution Books for the specific years of the questions.

Example 1. Let $p = 2^{3009}$, $q = 3^{2006}$ and $r = 5^{1003}$. Which of the following statements is true?

(A) $p < q < r$ (B) $p < r < q$ (C) $q < p < r$

(D) $r < p < q$ (E) $q < r < p$

(SMO Senior 2006)

Discussion of Example 1

The genre of the question in Example 1 is not unfamiliar to most upper secondary students, who are accustomed to arranging three or more given numbers in either ascending or descending. This genre of questions appears frequently in school-type examination questions for lower secondary students. What makes this question distinct from the other examination-type questions of this genre is that it involves extremely large numbers which are beyond the computation of a usual calculator. Thus the choice of the extremely large numbers forces students to resort to other means, possibly their school mathematics content knowledge (in this case, application of one of the rules of indices). This is a good illustration of the inconceivable "power" of mathematics over computational tools! The possible learning experience that teachers can introduce in this question is:

Comparing two numbers a^b and c^d.

- Students must observe that if either (i) $b > d$ and $a > c$; or (ii) $b < d$ and $a < c$, then it is immediately obvious which is the larger number is.
- If either (i) $b > d$ and $a < c$; or (ii) $b < d$ and $a > c$; the comparison is not easy. At this stage, students would have understood the problem (and the level of difficulty at this stage). This process of analytical reasoning is essential since in problem solving, one is expected to begin by attempting to understand the problem. Most often than not, students without perseverance would not even bother to attend to understand the problem and dismiss it as beyond their ability to solve this problem.

Looking for attributes in the numbers.

- Students could next examine the three given numbers p, q and r. Are there any attributes of these three numbers that might give the clue to how this problem can be solved? This provides the opportunity for the students to think of the attributes of the numbers in order to unlock the method of solving this question.
- While the bases 2, 3 and 5 are relatively small numbers, the powers 3009, 2006 and 1003 are much larger numbers with common factors.
- Students could link to their knowledge of school mathematics to attempt to proceed to solve this problem (for example, factorization and application of the rules of indices).

Expressing the three numbers with common power 1003

- At the next stage, students could express each of the numbers with the common power 1003 but different bases.
- For two numbers a^{1003} and b^{1003}, it is easy to compare the magnitude of the numbers now. It is important to emphasize that in the possible processes that students might move through, it is always noted that students' "resources" are equally important. While some younger students might have the wrong conception that problem solving is something esoteric and separate from school mathematics, the experience that students encounter in solving such competition questions tends

to show otherwise; "resources" in the form of mathematical content knowledge is equally important in problem solving.

Exploring alternative solution

- It would be good if students could look back at their solution. In this question, perhaps, students can explore any other alternative solution. For instance, if they are permitted to use a calculator, how could they obtain the answer immediately (of course, they could not punch the numbers into their calculator; they could explore the use of their knowledge of the properties of logarithm!).
- Students could even attempt to form some sort of generalization and communicate an "algorithm" of how to compare two numbers a^b and c^d for either the case (i) $b > d$ and $a < c$; or (ii) $b < d$ and $a > c$ (of course, this "algorithm" might not be easily applicable in general other than some clearly constructed questions).

Example 2. Suppose that $a_1, a_2, a_3, a_4,...$ is an arithmetic progression with $a_1 > 0$ and $3a_3 = 5a_{13}$. Find the value of n such that the sum of the first n terms of the arithmetic progression has the maximum value. (SMO Open 2013)

Discussion of Example 2

The phrasing of Example 2 is likely to be familiar for most average A-level students, who could easily recognize the mathematical content knowledge related to this question (Arithmetic Progression).

Trying to understand the Question in Example 2

- The issue of finding the "maximum value" of the sum of the first n terms of an arithmetic progression is a little puzzling on first thought for most students (impressionistically, the more terms you add, the larger the sum – so, shouldn't the maximum value of n be "infinity"?).
- After careful thinking, "maximum value" for the sum of an arithmetic progression could make sense in certain cases (that is, where the common difference is negative). Note that at this stage, the student is still trying to make sense of the question, without actually solving the question. It is crucial for students to start making sense of a mathematics problem. This process of attempting to make sense of the question by trying to analyse the situation based on content knowledge and "common sense" is an important part of problem solving.

Using available information and appropriate algebraic equations

- Students could apply their knowledge of the general term of an arithmetic sequence to make sense of the equation $3a_3 = 5a_{13}$ into a form that tells crucial information (that the sum of the 20^{th} and the 21^{st} term of the arithmetic progression is zero). Thus, the process of applying the correct formulae is still part of solving this problem.

Interpreting the information

- With the above information, students can make logical deduction to obtain the correct answer to this problem.

Exploring alternative solution

- It is expected that students are generally not comfortable with the above approach of deductive reasoning. Perhaps, students can explore a more mathematical way, e.g. using the formula for S_n, the sum of the first n terms of an arithmetic progression in terms of n and the first term a_1, which is a quadratic function of n. The problem reduces to one of finding the maximum value of a quadratic function.
- Students can even be invited to explore on how they could be able to modify this problem. For instance, if the arithmetic progression is replaced by a geometric progression, will this problem make sense? If not, how could we modify this problem to make it sensible?

Scaffolding

The illustrations above show how the mathematics competition questions could be used to provide the rich learning experience of students. However, one could not expect that the processes discussed above will spontaneously arise in students when they first encounter these problems (the more likely reaction of most students would be to give up solving the problems!). Teachers need to provide the essential scaffolding to engage their students in the process of problem solving! Scaffolding is especially required when students encounter a situation they find difficult to understand or a problem that they are unable to solve by their own "unassisted efforts" (Wood, Bruner & Ross, 1976).

Toh, Quek, Leong, Dindyal and Tay (2011) discuss the role of scaffolding offered by teachers in the mathematics classrooms when students come to a "dead end" in solving unfamiliar problems. Teachers must be reminded that the purpose of scaffolding is to engage the students in the problem solving processes rather than to provide direct hint or help on the correct method to solve the problem. Toh et al (2011) distinguished three levels of scaffolding: (1) general – emphasis on the general processes; (2) specific – emphasis on the method to solve the same type of problems; and (3) problem-specific – what to consider in solving that particular problem. Teachers should begin with the general scaffolding when students first ask for help; then move on to (2) and (3) if the students request for further assistance after being unable to solve the problem.

It should also be commented that scaffoldings at levels (1) and (2) should aim at the students' total engagement in the problem solving processes, in particular, the Polya's problem solving model. It is only when students experience repeated failure and frustration in solving the problem even after applying the entire problem solving processes that level (3) scaffolding be provided.

An example of a scaffolding scheme modified from Toh et al (2011) for handling mathematics competition questions and based on the Polya's model is shown below.

Level (1) scaffolding
Stage I: Understanding the problem a) Do you have difficulty understanding the statement of the problem? If yes, how did you overcome it? b) Do you think the question here makes sense? Which part does not make sense? Why do you think so? c) So finally, how does this question make sense to you?
Stage II: Devise a plan a) What is the mathematics content knowledge that you need to solve this question? Do you think you have learnt sufficiently well to solve this question? b) Specifically, which rule do you need to apply to solve this problem?
Stage III: Carry out the plan a) For each of the information or steps, write down the equations (or draw the diagrams) that you think is needed to solve the problems. b) Based on your equations (or diagrams), what can you conclude? c) Now complete solving the entire problem.
Step IV: Check and Expand a) Are you convinced that your solution is correct? Why or why not? b) Can you think of any alternative method to solve this problem? If yes, compare the two solutions of this problem. c) Write down at least one or two other problems that your method in solving this problem can be used.

Level (2) scaffolding, consisting of problem-specific prompts provided by teachers, is used when students are not able to continue with the problem solving processes based on level (1) scaffolding and when the latter have identified that they have the sufficient "resources" to solve the problem. It is crafted by teachers who have fully understood the rationale of the mathematical problem. Here a set of level (2) scaffolding for Example 1 is discussed, together with the anticipation of students' errors and misconceptions.

Level (2) scaffolding	**Students' likely response**	**Note to teachers**
• What do you observe about the three numbers p, q and r?	• The numbers are very big • The powers of the three numbers have common factor	• The rationale of the first prompt is to lead students to observe that the powers of the three numbers have a common factor 1003. However, teachers must be prepared to guide the students to observe this.
• [After observing that all the three powers have a common factor 1003] How will you represent these three numbers involving 1003?	• Different [incorrect] approach to perform this task	• It is very likely that students will perform this task incorrectly. Teachers can serve a constant reminder on the appropriate rule of indices to apply.
• [After expressing the three numbers in terms of 1003] How do you compare these three numbers?	• Students might not express the numbers in a form that is easy to compare, e.g. $p = (2^3)^{1003}$ and $q = (3^2)^{1003}$ and not able to compare	• Teacher provides the hint to simplify the numbers in the form a^{1003} directly.

It is instructional for the readers to develop a set of level (2) scaffolding for Example 2 above (and for other competition questions that they think are mathematically rich for their students).

Level (3) scaffolding involves giving students direct instruction in applying the correct procedures in getting to the correct solution. Whenever possible, teachers should avoid level (3) scaffolding if levels (1) and (2) scaffolding are sufficient.

Conclusion

This paper discusses the potential learning experiences that mathematics competition questions can provide for the general student population necessary for problem solving. Scaffolding based on Polya's problem solving model is also proposed for teachers in their mathematics classrooms. It is next necessary for teachers to select how other mathematics competition questions can be used to provide enriching learning experiences. However, it is essential that teachers are excited over solving these problems themselves (instead of dismissing these questions as only suitable for the elite few) before they can fire the enthusiasm in their students.

References

1. Bruner, Ross & Wood (1976). The Role of Tutoring in Problem Solving. *Journal of Child Psychology and Psychiatry*, Volume 17, Issue 2, pp. 89–100.
2. Cai, J., & Brook, M. (2006). Looking back in problem solving. *Mathematics Teaching Incorporating Micromath*, 196, 42 – 45.
3. Holton, D. (2010). A first step to Mathematical Olympiad problems. Singapore: World Scientific.
4. Ministry of Education (2013). Mathematics syllabus – Secondary. Singapore: Author.
5. Polya, G. (1945). How to solve it. Princeton: Princeton University Press.
6. Tay, T.S., To, W.K., Toh, T.L., & Wang, F. (2011). Singapore Mathematical Olympiads 2011. Singapore: Singapore Mathematical Society.
7. Toh, T.L., Quek, K.S., Leong, Y.H., Dindyal, J., Tay, E.G. (2011). Making Mathematics practical: An approach to problem solving. Singapore: World Scientific.

*The author is an Associate Professor at National Institute of Education,
Nanyang Technological University, Singapore.*